Curved-fiber pull-out model for nanocomposites. Part 2: Interfacial debonding and sliding

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\textbf{A B S T R A C T}

This paper is the second part in a series of works in which an analytical curved-fiber pull-out model for nanocomposites is proposed. The model includes the three stages of interface conditions—well-bonded, debonding, and sliding—involved in the entire pull-out process of a single curved fiber. In the first paper, the fiber and matrix are well-bonded, while in this second paper, the fiber and matrix are allowed to debond and slide, two relevant mechanisms in the later stages of pull-out. With either a constant or Coulomb friction interface, the pull-out model predicts higher pull-out forces as the fiber curvature increases, with zero fiber curvature (a straight fiber) producing the lowest pull-out forces. Fiber curvature effects are more pronounced, however, for the Coulomb friction model than the constant friction model because it considers radial compressive stresses at fiber/matrix interface. For the Coulomb friction model, two-dimensional finite element simulations are performed to test some of the model’s approximation. Results indicate reasonable agreement between the two.

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1. Introduction
1.1. Problem statement

Carbon nanotubes are one of the strongest materials available and consequently there is a great interest in carbon-nanotube-reinforced polymer matrix composites for structural applications. The number of studies focused on the stiffness and strength of this material system is large (Andrews et al., 2002; Breuer and Sundararaj, 2004; Buryachenko et al., 2005; Coleman et al., 2006; Valavala and Odegard, 2005). However a fundamental but less well evaluated property is their fracture behavior and toughness (Andrews and Weisenberger, 2004). Among the few studies performed to date, the effect of adding nanotubes into the polymer matrix on the composite toughness is inconclusive. Some report a relatively small increase in toughness (Gojny et al., 2005, 2004), some substantial losses in ductility and hence toughness (Moniruzzaman et al., 2006; Yasmin et al., 2006; Zheng et al., 2004), while others significant improvements in toughness (Blond et al., 2006; Chen et al., 2005; Dondero and Gorga, 2006; Fiedler et al., 2006; Yang et al., 2007).

In contrast, the toughness of conventional fiber composites has been studied extensively and the effects of reinforcement geometry, volume fraction, fiber/matrix interface, and fiber and matrix deformation behavior are relatively well understood. In these systems, one mechanism for increased toughness is fiber bridging. The harder it is for a growing crack to pull bridging fibers out of its wake, the higher the toughness of the composite.

While for the most part, the toughness and fracture behavior of nanotube-reinforced polymer matrix composites will share some of the same dependencies and features, the microstructure of these composites is unlike that of a conventional short fiber composite or continuous fiber composite. The aspect ratios of the nanotubes are...
higher than those of a continuous fiber composite and the nanotubes are typically randomly oriented, tangled together, and curved (Fig. 1). Therefore, more study is required for a better understanding of the deformation mechanisms governing the fracture behavior of nanotube composite material systems. One important fracture mechanism is nanotube pull-out. The load required to pull out a nanotube partially embedded in a matrix material is directly linked to the ability of nanotubes to bridge cracks in a composite.

As in conventional fiber pull-out, there are three stages in nanotube pull-out following the moment the load is applied to the fiber: the bonded stage, debonding stage, and sliding stage. In the first, the fiber and matrix are still well-bonded to each other. Under an applied force to pull it out, the fiber stretches and shear stresses are generated at the interface. As this pull-out force increases, so do the interfacial shear stresses. A critical point may be reached that initiates debonding of the interface. During interfacial debonding, part of the fiber moves along the debonded interface resisted by a friction force, while the rest of the fiber is still well-bonded to the matrix. When debonding extends over the entire interface area, sliding occurs. In this final stage, the entire fiber slides through the matrix resisted by frictional forces.

Debonding and sliding stages in nanotube pull-out have been studied by continuum mechanics and molecular mechanics approaches (Frankland et al., 2002; Frankland and Harik, 2003; Gou et al., 2004; Liao and Li, 2001; Lordi and Yao, 2000; Wagner, 2002; Wong et al., 2003; Xiao and Liao, 2004). Most of these models, however, ignore the debonding stage and only deal with the sliding stage. The molecular models further assume constant sliding. They all consider straight nanotubes. Although the effect of nanotube curvature on stiffness has been addressed previously (Bradshaw et al., 2003; Fisher et al., 2003), no work has been done to examine the influence of curvature on ductility and fracture toughness.

In this series of papers, we conduct a theoretical investigation on the curvature effect on nanotube pull-out. This effect is added to a shear lag model previously developed for the pull-out of straight fibers (Lawrence, 1972). Our first paper (Chen et al., 2009) constructed a new pull-out model for a fiber with constant curvature assuming that the fiber and matrix were well-bonded, applicable to stage 1. This second paper focuses on the formulation for the debonding and sliding stages. Although the curved-nanotube morphology is the motivation for modeling curved-fiber pull-out, our analytical model can be applied to matrices reinforced by curved fibers on both nano and conventional length scales.

The structure of this paper is as follows. First a brief review on conventional straight fiber pull-out models with debonding and sliding interfaces is given. Then the solution for single curved-fiber pull-out during the debonding and sliding stages is derived, assuming either constant friction or Coulomb friction at the interface. The latter is shown to be inherent in capturing the effects of fiber curvature. With Coulomb friction, the effects of fiber geometry and material properties on the pull-out curves of pull-out force $P_f$ versus pull-out displacement $\delta$ for curved fibers are further explored. Finally, results from a 2D finite element model are compared with the analytical model in the case of curved fibers with a Coulomb friction interface.

1.2. Review: previous straight fiber pull-out models for debonding and sliding stages

In conventional fiber-reinforced composites, fibers are usually straight, and therefore, most pull-out models for the debonding and sliding stages treat straight fibers (Cox, 1990; Faber et al., 1986; Freund, 1992; Gao et al., 1988; Grande et al., 1988; Hsueh, 1988, 1990a, 1992a; Hutchinson and Jensen, 1990; Kerans and Parthasarathy, 1991; Kim and Mai, 1998; Lawrence, 1972; Marshall, 1992; McCartney, 1989; Mumm and Faber, 1995; Nairn, 1997; Nairn and Wagner, 1996; Rosen, 1964; Sridhar et al., 2003; Takaku and Arridge, 1973; Tsai and Kim, 1996; Wu and Davies, 2005; Wu et al., 2000; Wu and Yu, 1994). For the debonding or sliding stages, either the debonded part of or the entire fiber moves relative to the matrix along the fiber/matrix interface resisted by a friction stress. In these models, this stress is usually constant (Brun and Singh, 1988; Deshmukh and Coyle, 1988; Goettler and Faber, 1988; Hutchinson and Jensen, 1990; Lawrence, 1972; Morscher et al., 1990) or governed by a Coulomb friction law under a certain residual radial stress (Cox, 1990; Freund, 1992; Gao et al., 1988; Hsueh, 1990a, 1991a; Hutchinson and Jensen, 1990; Kerans and Parthasarathy, 1991; Tsai and Kim, 1996).

Another important factor in the pull-out analysis is the debonding criterion. This criterion depends on whether interfacial debonding is treated as a force equilibrium problem or as a mode II fracture problem. In the former, a maximum shear stress is used as the debonding criterion (Beyerlein et al., 1998, 2003; Hsueh, 1988, 1990a,b, 1991b; Lawrence, 1972; Takaku and Arridge, 1973), while in the latter, a fracture energy criterion is applied (Gao et al., 1988; Hutchinson and Jensen, 1990; Wang, 1997). Experimental results for certain composite systems support the strength criterion (Beaumont and Aleszka, 1978; Watson and Clyne, 1992), while results for some other systems agree with the fracture energy criterion (Wells and Beau-
mont, 1985). Hsueh (1992b) suggested comparing experimental and theoretical results for dependences of both debonding criteria on material properties in order to decide which criterion is appropriate. A theoretical review and discussion on both criteria in application for fiber composite modeling can be found in Beyerlein and Phoenix (1997).

2. Analytical derivation of single curved-fiber model for debonding and sliding stages

Like our first paper of this series for the bonded stage (Chen et al., 2009), our analysis for the debonding and sliding stages is also based upon the Lawrence (Lawrence, 1972) model. Lawrence modeled both the bonded stage and debonding stage using a shear-lag approach. His model produced two types of debonding: stable or ‘progressive’ debonding and unstable or ‘catastrophic’ debonding. Before treating a curved fiber, we find it necessary to extend his model to include the sliding stage. With this, we are able to construct the entire \( P_f - \delta \) curve with all three stages for straight fiber pull-out. This straight fiber model is developed in Appendix.

For a curved fiber, three more modifications are made to the Lawrence model. The first two were also employed in the first paper of this two-part series. (1) As all shear lag models up to date, the Lawrence model is for a straight fiber. In our analysis it is modified to account for fiber curvature. (2) Lawrence assumed a free end at the fiber embedded end. Nanotubes are generally entangled as in Fig. 1 and therefore we use a non-zero stress at the fiber embedded end to account for this entanglement. (3) In the Lawrence model, the debonded part of the fiber was resisted by constant friction. In our curved-fiber model, we will also assume constant interface friction, but later show that this type of interface does not capture well the effects of fiber curvature on pull-out. A Coulomb friction interface is then applied and proven to be much more appropriate.

Fig. 2 shows the three stages in curved-fiber pull-out and the curvilinear system used in our analysis. The same fiber/matrix system and the same notation are used in this paper for stages II and III as in the analysis in our first paper (Chen et al., 2009) for stage I. \( R \) is the radius of fiber curvature, and \( r_f \) is fiber radius. We use a curvilinear system with tangential direction \( s \) and radial direction \( r \). \( s = 0 \) at the fiber embedded end. The angle characterizing the fiber geometry is \( z \). The original fiber embedded length \( L \) equals \( zR \). Our current model is valid for any \( z \) value between 0° and 180°. \( s_0 \) is the initial ‘free length’, i.e., the fiber length originally outside the matrix. \( P_f \) is the pull-out force at the fiber pulled end.

In the following analyses, all equations will be in normalized form to remove any unnecessary dependencies of the form of the solution on parameters. The normalization factor for length is the fiber radius \( r_f \). The factor for stress and moduli is the fiber Young’s modulus \( E_f \). Accordingly, the factor for force is \( \pi (r_f^4) E_f \). The asterisks indicate normalized values.

2.1. Constant friction model for interfacial shear stress in debonding and sliding stages

In this section, the interfacial shear stress governing the relative movement between fiber and matrix is modeled as constant friction. The entire pull-out curve is constructed and compared with the pull-out curve for a conventional straight fiber.

From the bonded stage solution (Chen et al., 2009), the maximum interfacial shear stress is shown to be

\[
\tau_{\text{imax}}^* = \frac{1}{2} \sqrt{T^* \coth (\sqrt{T^* L^*}) - \frac{\sigma_0^*}{\sinh (\sqrt{T^* L^*})}},
\]

where

\[
T^* = \frac{2G_m^*}{\ln \left( \frac{1 - \frac{\pi^2 s_0^2}{r_f^2}}{1 - \frac{\pi^2 s^2}{r_f^2}} \right)} \left[ 1 - \frac{1}{E_m^* r_f^{4\alpha\alpha}} \right].
\]

\( \sigma_0^* \) is fiber embedded end axial stress, \( G_m^* \) is the matrix shear modulus, \( E_m^* \) is the matrix Young’s modulus, and \( r_f^{4\alpha\alpha} \) is the matrix radius. \( T^* \) is an important variable which packages the effects of radius of curvature \( R^* \). The maximum interfacial shear stress increases as \( P_f \) is increased. According to the maximum shear stress criterion, debonding begins when \( \tau_{\text{imax}}^* \geq \tau_i^* \), where \( \tau_i^* \) is the critical shear stress.

2.1.1. Debonding stage

In this stage, part of the fiber has debonded from the matrix. Fig. 3 shows the stress equilibrium for the debonded and well-bonded parts, where \( L - z \) is the debonded

\[ \text{Stage I:} \quad \text{bonded} \]
\[ \text{Stage II:} \quad \text{debonding} \]
\[ \text{Stage III:} \quad \text{fully debonded/ sliding} \]

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Fig. 2. Three stages in pull-out of a single curved fiber. The figure for stage I also illustrates the model geometry. The 2D curved fiber has a constant radius of curvature \( R \) and circular cross-section of radius \( r_f \) in curvilinear coordinate system \( s \) and \( r \). \( s_0 \) is the free length of the fiber initially not embedded in the matrix.
length, and \( z \) is the bonded length. In Fig. 3, \( P_f' = \pi (r')^2 \sigma_f \) is the fiber axial load at the boundary between these two parts, \( P_f = \pi (r')^2 \sigma_i \) is the applied pull-out load, \( \sigma_i \) is the corresponding fiber axial stress \( \sigma_i(s = L) \), and \( \sigma_0 \) is the stress at the fiber embedded end. The debonded part suffers a constant interfacial friction stress, denoted as \( \tau_{ic} \), which opposes the pull-out force as shown in Fig. 3(a). The interfacial shear stress for the bonded part, \( \tau_s \), acts in the opposite direction of the pull-out force as shown in Fig. 3(b). According to equilibrium, we have for the debonded part (Fig. 3(a)):

\[
P_f' - P_f'' = 2\tau_{ic}(L' - z').
\]

(3)

In order to calculate the total fiber displacement, we need an expression for the axial stress in the fiber, \( \sigma_i(s) \) which can be obtained as follows:

\[
\sigma_i(s) = P_f' + 2\tau_{ic}(s' - z'), \quad s' \in (z', L').
\]

(4)

For the well-bonded part (Fig. 3(b)), we use the solution for the bonded stage from the first paper (Chen et al., 2009), where the interfacial shear stress is:

\[
\tau_s = \frac{1}{2} \left[ \left( \frac{P_f'}{\sinh(\sqrt{T}z')} - \sigma_0 \coth(\sqrt{T}z') \right) \cosh(\sqrt{T}z') + \sigma_0 \sinh(\sqrt{T}z') \right], \quad s' \in (0, z').
\]

(5)

and the fiber axial stress is:

\[
\sigma_i(s) = \left( \frac{P_f'}{\sinh(\sqrt{T}z')} - \sigma_0 \coth(\sqrt{T}z') \right) \sinh(\sqrt{T}s') + \sigma_0 \cosh(\sqrt{T}s'), \quad s' \in (0, z').
\]

(6)

At the boundary between the bonded and debonded parts, \( s' = z' \), \( \tau_s = \tau_i \), and the fiber axial force \( P_f' \) in Eqs. (3) and (5) are equal. Applying these conditions, we have

\[
P_f' = 2\tau_{ic}(L' - z') + 2\tau_{ic} \tanh(\sqrt{T}z') + \frac{\sigma_0}{\cosh(\sqrt{T}z')}.
\]

(7)

From Eq. (7), we find that \( P_f' \) is a function of \( z' \), and it reaches a maximum value \( P_{f,max} \) when \( \frac{dP_f'}{dz'} = 0 \) at \( z' = z^{0*} \), where

\[
z^{0*} = \frac{1}{\sqrt{T}} \sinh^{-1} \left( \frac{\sqrt{\frac{\sigma_0}{T}}}{4\tau_{ic} + 4\tau_{ic} \coth(\tau_{ic} \tau_i) - \sigma_0 \sqrt{T}} \right).
\]

(8)

If \( z^{0*} > L' \), debonding is catastrophic, which means that the current pull-out force is large enough to debond the rest of the fiber. If \( z^{0*} < L' \), progressive debonding occurs, which means that the pull-out force needs to be increased in order to propagate further the debond. Through \( z^{0*} \), the stability of debond propagation is predicted to depend on \( R^*, \tau_{ic}, \tau_i, \sigma_0, \) and \( L' \).

The corresponding peak loads \( P_{f,max} \) for unstable and stable debond propagation are, respectively,

\[
P_{f,max} = \begin{cases} 
2\left( \tanh(\sqrt{T}L') + \frac{\sigma_0}{\cosh(\sqrt{T}L')} \right), & \text{if } z^{0*} > L' \\
2\tau_{ic}(L' - z^{0*}) + \tau_{ic} \tanh(\sqrt{T}z^{0*}) + \frac{\sigma_0}{\cosh(\sqrt{T}z^{0*})}, & \text{if } z^{0*} < L'.
\end{cases}
\]

(9)

Fig. 4 shows example curves of \( P_f' \) versus the debond length \( L' - z' \) for these two situations. In both, \( R^* = 70 \) and \( z^{0*} = 20.8 \). In Fig. 4(a), \( L' = 33.3 \) and is greater than \( z^{0*} \). Accordingly, debond propagation is initially stable, meaning that with increasing \( L' - z' \), \( P_f' \) increases. Catastrophic debonding is delayed until \( z' \) drops below \( z^{0*} \). In Fig. 4(b) \( L' = 19 \) and is less than \( z^{0*} \). So catastrophic debonding happens instantaneously.

The fiber displacement \( \delta^{[II]} \) (superscript (II) denotes stage II) can be written as a function of \( z' \) using Eqs. (3), (4), (6), and (7).
\[ \delta^{(\text{II})} = \int_0^\delta \frac{\sigma_i^0}{\sinh(\sqrt{T}z') - 1} \, ds' + \frac{P_f^i s^0}{\sqrt{T}} \]

After the entire fiber is debonded from the matrix, i.e., \( z = 0 \), the sliding stage begins and the entire embedded portion of the fiber is resisted by frictional forces. During this stage, the fiber undergoes a rigid body sliding displacement equal to \( L - s^{(\text{III})} \), where \( s^{(\text{III})} \) is the embedded fiber length, and elastic elongation. Stage III ends when the fiber is completely pulled out, i.e., \( s^{(\text{III})} = 0 \).

Fig. 5 shows the stress equilibrium in the fiber while it is sliding. As in the debonding stage, in order to calculate the total fiber displacement, we will need an expression for the axial stress in the fiber, \( \sigma'_0(s) \). From equilibrium, we have

\[ P_f^i = 2\tau_i^c s^{(\text{III})} \]

\[ \sigma'_0(s') = 2\tau_i^c s' \]

In sliding, fiber displacement is composed of three parts: elastic elongation of the embedded part, elastic elongation of the free length \( s_0^i \), and the rigid sliding displacement:

\[ \delta^{(\text{III})} = \int_0^{s^{(\text{III})}} \varepsilon_i^d ds' + P_f^i s_0^i + (L' - s^{(\text{III})}) \]

\[ = \frac{P_f^i}{4\tau_i^c} + P_f^i s_0^i + L' - 2\tau_i^c. \]

Now we have the three \( P_f^i - \delta' \) relationships needed to model the entire pull-out process, i.e., Eq. (10) for the debonding stage \( P_f^i - \delta^{(\text{II})} \), Eq. (13) for the sliding stage \( P_f^i - \delta^{(\text{III})} \), and the equation for the bonded stage \( P_f^i - \delta^{(\text{I})} \) as shown below from the first paper (Chen et al., 2009):

\[ \delta^{(\text{I})} = \frac{1}{\sqrt{T}} \left( \frac{P_f^i \coth(\sqrt{T}L')}{\sinh(\sqrt{T}L')} - \frac{P_f^i}{\sqrt{T}L'} - \frac{\sigma_0^c}{\sinh(\sqrt{T}L')} \right) \]

\[ + \frac{\sigma_0^c}{\sqrt{T}L'} + P_f^i s_0^i. \]

The entire \( P_f^i - \delta' \) curve is calculated using the following strategy. Starting from the bonded stage, we increase \( \delta' \) from zero and calculate the corresponding pull-out force \( P_f^i \) using Eq. (14) and the corresponding maximum interfacial shear stress \( \tau_i^c(L') \). If \( \tau_i^c(L') > \tau_i^c \), the debonding stage begins. In the debonding stage, we first obtain \( z^{(\text{III})} \) from Eq. (8) and compare it with the fiber embedded length \( L^* \). If \( z^{(\text{III})} > L^* \), catastrophic debonding occurs and the sliding stage follows immediately. Otherwise, progressive debonding occurs. In progressive debonding case as \( \delta' \) increases, Eq. (10) is used for the bonded length \( z^* \) and Eq. (7) for \( P_f^i \). The increments continue until either the entire fiber has debonded, i.e., \( z^* < 0 \), or \( z^* < z^{(\text{III})} \), the latter of which means during this increment, stable debonding becomes catastrophic. The occurrence of either marks the beginning of the sliding stage and the corresponding \( P_f^i \) at this point, calculated from Eq. (9), is the peak value over the entire \( P_f^i - \delta' \) curve. As \( \delta' \) increases further in the sliding stage Eq. (13) is applied for \( P_f^i \) and Eq. (11) is used for the fiber embedded length \( s^{(\text{III})} \). The increments continue until \( s^{(\text{III})} \) approaches zero, meaning the entire fiber has been pulled out. This is the end point of our \( P_f^i - \delta' \) curve.

2.1.3. Comparison with the straight fiber case

As mentioned previously, we build in Appendix the analogous solution for the pull-out of a straight fiber starting from the model of Lawrence (Lawrence, 1972). It is summarized below, where \( x \) denotes the fiber axial direction and \( L^{(\text{III})} \) denotes the remaining embedded fiber length in the sliding stage. (\( L^{(\text{III})} \) corresponds to \( s^{(\text{III})} \) in the curved-fiber solution.)

During debonding, we have:

\[ P' = 2\tau_i^c (L' - z') + \frac{2\tau_i^c}{\sqrt{Q'}} \tanh \left( \sqrt{Q'} z' \right) + \frac{\sigma_0^c}{\cosh \left( \sqrt{Q'} z' \right)}. \]

\[ x' = \frac{1}{\sqrt{Q'}} \sinh^{-1} \frac{\sqrt{Q'} z' - 16\tau_i^c (\tau_i^c - \tau_i^c) - \sigma_0^c / \sqrt{Q'}}{4\tau_i^c Q'}. \]

\[ P_{\text{max}}' = \frac{2\tau_i^c (L' - z')}{\sqrt{Q'}} + \frac{2\tau_i^c}{\sqrt{Q'}} \tanh \left( \sqrt{Q'} z' \right) + \frac{\sigma_0^c}{\cosh \left( \sqrt{Q'} z' \right)} \]

\[ \delta^{(\text{III})} = \left( \frac{\cosh \left( \sqrt{Q'} z' \right)}{\sinh \left( \sqrt{Q'} z' \right)} - \frac{1}{\sqrt{Q'}} \right) \left( \frac{2\tau_i^c \tanh \left( \sqrt{Q'} z' \right)}{\sqrt{Q'}} + \frac{\sigma_0^c}{\cosh \left( \sqrt{Q'} z' \right)} \right) \]

\[ + \left( \frac{L' - z'}{\sqrt{Q'}} \right) \left( \frac{2\tau_i^c \tanh \left( \sqrt{Q'} z' \right)}{\sqrt{Q'}} + \frac{\sigma_0^c}{\cosh \left( \sqrt{Q'} z' \right)} + \tau_i^c (L' - z')^2 \right) \]

\[ + \left( \frac{L' - z'}{\sqrt{Q'}} \right) \left( \frac{2\tau_i^c \tanh \left( \sqrt{Q'} z' \right)}{\sqrt{Q'}} + \frac{\sigma_0^c}{\cosh \left( \sqrt{Q'} z' \right)} + 2\tau_i^c (L' - z') \right). \]

Fig. 5. Stress equilibrium of an embedded fiber during sliding. \( s^{(\text{III})} \) is the embedded fiber length and \( L - s^{(\text{III})} \) is the rigid body sliding displacement.
where

\[ Q' = \frac{2G_m'}{\ln r_m^* \left(1 - \frac{1}{r_m^* E_m'}\right)} \]  

During sliding,

\[ \sigma'(x) = \frac{P' x'}{L^{(m)*}}, \]  

\[ \delta^{(m)*} = \frac{P'^2}{4\tau_{ic}} + \frac{P' T_0}{L} + \frac{P'}{2\tau_{ic}}. \]  

\[ Q' \text{ in Eq. (19) packages the influence of the matrix mechanical properties and fiber geometry. It corresponds to } T^* \text{ in the curved-fiber solution. Recall that} \]

\[ T^* = \frac{2G'_m}{\ln \left(1 - \frac{r'_m}{r_m^*}\right) \left(1 - \frac{1}{r_m^* E_m'}\right)} \]  

when \( R^* \) goes to infinity, \( T^* \) approaches \( T^* \), and Eqs. (7)–(10) and Eqs. (12), (13), converge to those for a straight fiber, Eqs. (15)–(18) and Eqs. (20), (21), respectively.

Fig. 6 compares the entire \( P'_f - \delta' \) curves for a fiber with \( R^* = 42 \), \( R^* = 70 \), and \( R^* = \infty \) (straight fiber case). Table 1 lists the remaining model parameters. As shown, when \( R^* \) decreases, \( P'_f \max \) increases. Note how the curved-fiber case approaches the straight fiber case as \( R^* \) increases.

In spite of the wide range of \( R^* \), its effect on the \( P'_f - \delta' \) curves is small. The predicted insensitivity to curvature is likely an artifact of neglecting the interfacial radial compressive stress, which can be substantial in the curved-fiber case, and its effect on friction. For the calculations in Fig. 6, the same constant interfacial friction stress \( \tau_{ic} \) is assigned to both the straight and curved fibers. However, it is expected that the interfacial friction stress is much larger for the curved fiber than for the straight fiber due to a non-negligible radial compressive stress from the matrix along the inner surface of the curved fiber. In the next section, we include radial compressive stresses into our model as a function of \( R^* \), and their effect on interface friction through a Coulomb friction law.

2.2. Coulomb friction model for interfacial shear stress in debonding and sliding stages

2.2.1. Debonding stage

As in the constant friction model, for the debonding stage, we consider a bonded length \( z \) and debonded length \( L - z \), but in this case, the \( L - z \) portion is resisted by an interfacial friction stress governed by Coulomb friction with coefficient \( \mu \) and the interfacial friction will depend on \( s \).

The solution involves first finding a governing equation in \( \sigma_r(s) \) within the debonded portion and then using it to calculate \( P_f \). Fig. 7 shows the stresses acting on the debonded portion including \( \sigma_r(s), \) Coulomb friction stress \( \tau_c, \) radial compression stress \( \tau_p \) and the debonding stage is as a function of \( R^* \), and their effect on interface friction through a Coulomb friction law.

\[ \tau_{ic} = \frac{\tau_c^*}{2} + \tau_0. \]  

From the stress equilibrium of the debonded fiber portion in Fig. 7, we have in the \( s \)-direction:

\[ -d\sigma_r^* \cos \frac{d\theta}{2} + \tau_c^* R' d\theta + \tau_0^* 2R' d\theta = 0 \]  

and in the \( r \)-direction:

\[ \int_0^\pi \sigma_r^* dz R' \sin z - \left(2\sigma_r^* \sin \frac{d\theta}{2} + d\sigma_r^* \sin \frac{d\theta}{2}\right) \pi \]  

\[ = 0. \]  

The Coulomb friction law is given by:

\[ \tau_r^* = \mu \sigma_r^*. \]  

As we will show, \( \sigma_r \) and thus \( \tau_c \), only result from fiber curvature. As the fiber curvature \( 1/R \) approaches zero, this pull-out model assuming Coulomb friction will degenerate

\[ \tau_{ic} = \frac{\tau_c^*}{2} + \tau_0. \]
In the debonded portion of the fiber, we have the axial stress
\[ \sigma_i(z^*) = \frac{2\tau_i^*}{\sqrt{T}} \tanh \left( \sqrt{T}z^* \right) + \frac{\sigma_0}{\cosh \left( \sqrt{T}z^* \right)} + \frac{4\tau_i R}{\mu \pi} e^{\frac{z^*}{R}}. \]

Substituting this into Eq. (28), we have the axial stress in the debonded portion of the fiber:
\[ \sigma_i = \frac{4\tau_i R}{\mu \pi} \left( \frac{2\tau_i^*}{\sqrt{T}} \tanh \left( \sqrt{T}z^* \right) + \frac{\sigma_0}{\cosh \left( \sqrt{T}z^* \right)} + \frac{4\tau_i R}{\mu \pi} \right) e^{\frac{z^*}{R}}. \]

Substituting Eq. (31) into Eqs. (23)–(26), we obtain the interfacial shear stress exerted on the debonded portion \( \tau^* \) and \( \tau_{\text{eff}} \):
\[ \tau^* = \frac{\mu \pi}{2R} \left[ \frac{2\tau_i^*}{\sqrt{T}} \tanh \left( \sqrt{T}z^* \right) + \frac{\sigma_0}{\cosh \left( \sqrt{T}z^* \right)} + \frac{4\tau_i R}{\mu \pi} \right] e^{\frac{z^*}{R}}. \]
\[ \tau_{\text{eff}} = \frac{\mu \pi}{4R} \left[ \frac{2\tau_i^*}{\sqrt{T}} \tanh \left( \sqrt{T}z^* \right) + \frac{\sigma_0}{\cosh \left( \sqrt{T}z^* \right)} + \frac{4\tau_i R}{\mu \pi} \right] e^{\frac{z^*}{R}}. \]

With the boundary condition at the pulled end, i.e., \( P_f = \sigma_i \) at \( z^* = L^* \),
\[ P_f = -\frac{4\tau_i R}{\mu \pi} \left( \frac{2\tau_i^*}{\sqrt{T}} \tanh \left( \sqrt{T}z^* \right) + \frac{\sigma_0}{\cosh \left( \sqrt{T}z^* \right)} + \frac{4\tau_i R}{\mu \pi} \right) e^{\frac{z^*}{R}}. \]

As in the constant friction case, \( P_f \) reaches its peak value at \( z^* = z^{\text{peak}} \) and depending on \( z^{\text{peak}} \) and \( L^* \), there are two debonding situations, stable and catastrophic debonding. Example \( P_f \) versus \( L^* - z^* \) curves for \( R^* = 21.2 \) and \( z^{\text{peak}} = 22.3 \) are given in Fig. 9(a) when \( z^{\text{peak}} < L^* = 33.3 \) and debonding is stable and in Fig. 9(b) when \( z^{\text{peak}} > L^* = 15 \) and debonding is unstable. In the former, Fig. 9(a), \( P_f \) increases as the debond zone propagates until \( z^* \) reduces below \( z^{\text{peak}} \) and debonding becomes catastrophic.

Using Eq. (6) and Eq. (31), the pull-out displacement in the debonding stage, \( \delta^{(\text{III})} \), is:
\[ \delta^{(\text{III})} = \int_0^{L^*} \tau_{\text{eff}} ds + \int_{z^*}^{L^*} \tau^* ds + P_f s_0 \]
\[ = \frac{1}{\sqrt{T}} \left[ \frac{\delta^i(z^*) + \sigma_0}{\cos \left( \sqrt{T}z^* \right)} \right] \left[ \coth \left( \sqrt{T}z^* \right) - \frac{1}{\sinh \left( \sqrt{T}z^* \right)} \right] \]
\[ + P_f s_0 - \frac{4\tau_i R}{\mu \pi} \left( \frac{2\tau_i^*}{\sqrt{T}} \tanh \left( \sqrt{T}z^* \right) \right) \frac{2R}{\mu \pi} \left( e^{\frac{z^*}{R}} - 1 \right), \]

where \( P_f \) is given in Eq. (34) and \( \sigma_i(z^*) \) is given in Eq. (29). Note that \( \delta^{(\text{III})} \) and its associated pull-out force \( P_f \) can be related through \( z^* \).

### 2.2.2. Sliding stage

In the beginning of sliding stage, the fiber and matrix have already debonded from each other, which means that during the sliding stage, the fiber embedded end is free. Therefore, for the sliding stage, we can use the same model as the constant friction model, \( \tau_0 \). Therefore \( \tau_0 \) can be used to account for surface roughness and material interface friction.

Making \( d\theta \) infinitesimally small, from Eqs. (24)–(26), we have:
\[ \frac{d\sigma_i}{ds} = \frac{\pi \mu \theta}{2} \sigma_i + 2\tau_0 \]
whose general solution is
\[ \sigma_i = -\frac{4\tau_i R}{\mu \pi} + Ae^{\frac{z^*}{R}}, \quad s^* \in (z^*, L^*). \]

The constant \( A \) in Eq. (28) can be determined by equating \( \sigma_i \) in Eq. (28) with \( \sigma_i^* \) in the bonded portion at \( s^* = z^* \), which is \( \text{(Chen et al., 2009)} \):
\[ \sigma_i(z^*) = \frac{2\tau_i^*}{\sqrt{T}} \tanh \left( \sqrt{T}z^* \right) + \frac{\sigma_0}{\cosh \left( \sqrt{T}z^* \right)} + \frac{4\tau_i R}{\mu \pi} e^{\frac{z^*}{R}}. \]

Substituting Eq. (30) into Eq. (28), we have the axial stress in the debonded portion of the fiber:
\[ \sigma_i = \frac{4\tau_i R}{\mu \pi} \left( \frac{2\tau_i^*}{\sqrt{T}} \tanh \left( \sqrt{T}z^* \right) + \frac{\sigma_0}{\cosh \left( \sqrt{T}z^* \right)} + \frac{4\tau_i R}{\mu \pi} \right) e^{\frac{z^*}{R}}. \]
as in the debonding stage but with a free embedded end boundary condition. In the general solution for $\sigma_0'$ in Eq. (28), we have

$$A = \frac{4\sigma_0'R}{\mu i}. \tag{36}$$

The fiber axial stress $\sigma_0'$ and the interfacial shear stresses $\tau_0'$ and $\tau_{eff}^*$ become

$$\sigma_0' = \frac{4\sigma_0'R}{\mu i} \left( 1 + 2e^{\frac{R}{\mu i}} \right). \tag{37}$$

$$\tau_0' = -2\sigma_0' + 2\tau_0 e^{\frac{R}{\mu i}}. \tag{38}$$

$$\tau_{eff}^* = \sigma_0 e^{\frac{R}{\mu i}}. \tag{39}$$

In Eq. (39), $\tau_{eff}^*$ increases exponentially from $s^* = 0$ to $s^* = L^*$ at a rate which decreases as $R^*$ increases, and equals the constant $\tau_0^*$ when $R^*$ is infinity. The exponential form of $\tau_{eff}^*$ distinguishes the Coulomb friction model from the constant one. This difference is illustrated in Fig. 10 which compares a typical $\tau_{eff}^*$ profile predicted from (a) the Coulomb friction model and the (b) constant friction model with $\tau_{eff}^*$. As shown in Fig. 10(a), the maximum $\tau_{eff}^*$ is reached at $s^* = L^*$. It may be expected, however, that the maximum $\tau_{eff}^*$ would be achieved somewhere in the middle of a fiber like when a fiber slides over a rigid pulley. The difference in our case is attributed to the applied boundary conditions. At the embedded end, zero $\sigma_0'$ leads to zero $\tau_0'$ and thus zero $\tau_{eff}^*$, reducing to $\tau_0^*$. At the point where the fiber leaves the matrix, the large axial stress leads to high pressures and hence high $\tau_{eff}^*$ values. Evaluating $\sigma_0'$ at $s^* = s^{(III)}_0$ yields $P_f^*$ associated with sliding:

$$P_f^* = \frac{4\sigma_0'R}{\mu i} + \frac{4\tau_0'R}{\mu i} e^{\frac{2R}{\mu i}}. \tag{40}$$

The fiber displacement associated with sliding, $\delta^{(III)}$, is composed of three parts: elastic elongation of the embedded part, elastic elongation of the free length $s_0^*$, and the rigid sliding displacement, $L^* - s^{(III)}_0$.

$$\delta^{(III)} = \int_0^{s_0^*} \varepsilon ds' + P_f s_0^* + (L^* - s^{(III)}_0).$$

$$\delta^{(III)} = \frac{4\sigma_0'R}{\mu i} e^{\frac{2R}{\mu i}} + \frac{2\tau_0'}{\mu i} \left( e^{\frac{2R}{\mu i}} - 1 \right) + P_f s_0^* + (L^* - s^{(III)}_0). \tag{41}$$

Note that $\delta^{(III)}$ and $P_f^*$ can be related through $s^{(III)}_0$. Using the same method as described before in Section (2.1.2), the complete pull-out curves can be calculated.

Fig. 11 shows the $P_f - \delta$ curves for two curved fibers with $R^* = 42$ and $R^* = 70$, and for a straight fiber ($R^* = \infty$), all with Coulomb friction interfaces. Table 2 lists the other parameters used. As $R^*$ increases, $P_{f,max}$ and the strain at which it is achieved decrease. Fibers with more curvature require higher loads and displacements to pull out.

Fig. 12 compares the $P_f - \delta$ curves between the constant friction and Coulomb friction interfaces for the same curved fiber ($R^* = 42$) and value of $\tau_0^*$. The Coulomb friction $\mu$ is 0.2 and the other parameters are those in Table 2. As shown, the $P_f - \delta$ curve for the curved fiber with Coulomb friction has the largest $P_{f,max}$ and during sliding requires the highest $P_f$. Clearly, accounting for the effect of interfacial radial compressive stresses on interface friction via Coulomb friction enhances the effect of fiber curvature in pull-out.
Comparing Figs. 11 and 12 shows that when the Coulomb friction model is applied to a straight fiber, it degenerates to the same result as that obtained from a constant friction model with $\tau_0$. Therefore, the $R^* = \infty$ case is the same in the Coulomb friction and constant friction models, the solution of which has already been derived in Appendix and summarized in Section 2.1.3.

2.2.3. Parametric study

Using the pull-out model with the Coulomb friction law we examine the effects of $R^*$, $L^*$, $\sigma_0$, $\tau_0$, and $\mu$ on pull-out behavior. We vary only one of these parameters within a physically reasonable range while the others are fixed at their base values: 70, 65, $10^{-9}$, $2 \times 10^{-5}$, and 0.2, respectively. Illustrated in Fig. 13 are (a) three $R^*$ values with constant $L^*$ and (b) three $L^*$ values with constant $R^*$. Note that segment lengths of constant curvature, $L^*$, will be much smaller than the total nanotube length, which can typically range from several hundreds up to several thousands and contain many curve segments pieced together from end

Fig. 13. Schematics showing different fiber geometries used in the parametric study: (a) fibers with different $R^*$ ($42$, 70, infinity) and (b) fibers with different $L^*$ ($33$, 65, 100).

Fig. 14. Results from the analytical curved-fiber pull-out model showing the effects on the normalized $P_f/C_0$ curves by changing following parameters from their base values (see text): (a) radius of curvature $R^*$, (b) fiber length $L^*$, (c) fiber embedded end stress $\sigma_0$, (d) constant interfacial shear stress $\tau_0$, and (e) friction coefficient $\mu$. Note all plots are on different scales in order to highlight the important features in each case.
to end. The remaining parameters are chosen to represent a nanotube/polymer system: $E_m/C_3 m = 1E - 3, G_m/C_3 m = 5E - 4, \tau_s/C_3 s = 3.5e - 5, r_0/C_3 m = 40$.

Fig. 14(a) shows the pull-out curves for $R^* = 42, 70, \text{and } \infty$. As shown, the smaller $R^*$, the larger $P_{f,max}$ and the larger the area under the pull-out curve, which is closely related to the energy required to pull out the fiber. As $R^*$ increases, the $P_f - L$ curve converges to that for a straight fiber ($R^* = \infty$), in which we have the smallest $P_{f,max}$ and the lowest toughness.

Fig. 14(b) shows the effect of changing $L^*$. As expected, it takes a larger force and more energy to pull out a longer fiber. The discontinuity in slope on the rising part of the curves indicates the onset of debonding stage. A longer fiber has a longer debonding stage and is less susceptible to catastrophic debonding. Of course this conclusion assumes that the fiber does not fracture during pull-out; a fracture during pull-out would shorten it.

Fig. 14(c) shows that a larger $\sigma_0$ leads to a higher $P_{f,max}$ and higher system toughness. A larger $\sigma_0$ could represent, for instance, more nanotube entanglement. In this enlarged figure, the discontinuity in slope on the rising part of the curves is more obvious. It clearly indicates the onset of debonding stage.

In Fig. 14(d) and (e) the two parameters governing interface friction conditions, $\tau_s$ and $\mu$, are changed independently. Increases in either $\tau_s$ or $\mu$ lead to increases in the maximum pull-out force and composite interface toughness. Moreover, a smaller $\tau_s$ or $\mu$ leads to a shorter debonding stage, indicating the likely occurrence of catastrophic debonding for fibers with weak interface friction. Same as in Fig. 14(c), there is an obvious discontinuity in slope in Fig. 14(e), indicating the beginning of the debonding stage. To summarize, we find that $L^*$ and $R^*$ have the largest impact on pull-out curves, followed by $\mu, \tau_s$ and last $\sigma_0$.

It is worthwhile pointing out that parameters used in our formulation could be obtained from experimental pull-out curves. For example, we have the value of $P^*$ at the beginning of the debonding stage and apply Eq. (28) in Part 1 (Chen et al., 2009), that is

$$\tau_s = \tau_{f,max} = \frac{1}{2} \sqrt{Q^* \left( P \coth(\sqrt{Q^* L}) - \frac{\sigma_0}{\sinh(\sqrt{Q^* L})} \right)},$$

where $Q^* = \frac{2G_m}{\ln r_m^*} \left( 1 - \frac{1}{r_m^2 E_m} \right)$.

then the critical shear stress $\tau_s$ can be calculated. Other parameters can also be determined from experimental data.

3. 2D finite element model for sliding stage

In the derivation employing a Coulomb friction interface law, the matrix is treated as a rigid body which only supports the fiber and does not deform. The whole system is then approximated as a fiber debonding and sliding along a rigid frictional pulley. To check this approximation,

![Fig. 15. The 2D FE model for curved-fiber pull-out with boundary conditions. First, pressure is applied at matrix outer surface and second, a uniform displacement is added to the fiber pulled end. The green mesh is the matrix and the red one is the fiber. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this paper.]

![Fig. 16. Stress distribution calculated from the 2D FE model during pull-out for case 2: (a) stress distribution for the entire system and (b) stress distribution for the fiber.]}
model predictions are compared against those from a finite element (FE) model with a deformable matrix, developed using ABAQUS/CAE. FE simulations are performed for various cases of $\mu$ and $R$ and the interfacial pressure distributions generated during sliding are compared with those predicted from the analytical model. A 2D plane strain model is used instead of 3D one for easy convergence and computational efficiency, and based on the expectation that the results calculated from 2D and 3D will qualitatively be similar. Because both debonding and sliding utilize the same assumptions for the interfacial shear stress, it is sufficient to conduct this comparison in sliding only.

In the sliding stage, the entire fiber moves relative to the matrix. Hence the fiber and matrix can be modeled as two bodies, each meshed with four-node plane strain elements. The matrix is constructed as one piece so that the upper and lower matrix portions are connected to each other by a layer of matrix beneath the fiber (shown in Fig. 15 and hidden in Fig. 16(a)). As in the analytical model, the fibers have a constant $R$. The matrix only interacts with the fiber through the contact surfaces, defined a priori in ABAQUS. The shear stresses along the contact surfaces are governed by Coulomb friction with coefficient $\mu$.

There are two steps in our finite element simulation. In the first step, pressure is applied at matrix outer surface to achieve an initial local compressive stress (or pressure $P_0$) at fiber/matrix interfaces (Fig. 15). The matrix inner surface is fully fixed, the matrix and fiber embedded ends are fixed in the 2-direction, and the matrix face at the pulled end is fixed in the 1-direction.\textsuperscript{1} As shown in Fig. 15, the matrix is constructed a little longer than the fiber at the pulled end to eliminate some artificial boundary effects. In the second step, the embedded end is released and the fiber is pulled out by a uniform displacement at the pulled end.

In the FE simulation, we consider five cases: In cases 1–3, the fiber is curved with $R = 1.02E-02$ m, $2.03E-2$ m, and $3.04E-2$ m, respectively, with $\mu = 0.3$. In cases 4 and 5, we consider case 3 ($R = 3.04E-02$ m) but with $\mu = 0.2$ and 0.4, respectively. The fiber radius $r_f$ is changed with $R$ to keep the aspect ratio similar in all cases. For $R = 1.02E-03$ m, $r_f = 7.50E-4$ m; for $R = 2.03E-2$ m, $r_f = 1.50E-3$ m; and for $R = 3.04E-2$ m, $r_f = 1.54E-3$ m. Table 3 lists the properties of the fiber, and matrix used in the FE analyses, and the pressure applied at the matrix outer surface.

In all cases, after the first FE step, the resulting pressure $P_0$ is reasonably uniform along the interface in $s$, with the exception of a spike at the pulled end. Due to the fiber curvature, $P_0$ is higher on the outer interface than inner interface. Therefore the initial interfacial shear stress associated with $P_0$ is not strictly a material property but one that also depends on curvature. The analytical model, on the other hand, assumes that the initial $\tau_0$ is constant.

After the second FE step, the interfacial pressures predicted from both models are compared. With a Coulomb friction interface, comparing the interfacial pressure is equivalent to comparing the interfacial shear stress. In our analytical model the average interfacial pressure in the lower half of the fiber surface during sliding can be calculated from Eq. (38):

\[
\frac{(\tau_c + \tau_0)}{\mu} = \left(\frac{\tau_c + \tau_0}{\mu} + \frac{P_0}{\mu} - 1\right).
\]

From the numerical model, $\tau_0/\mu$ can then be replaced by $P_0$.

\[
\frac{(\tau_s + \tau_0)}{\mu} = \left(\frac{\tau_s + \tau_0}{\mu} + P_0\right).
\]

where the maximum value is achieved at the pulled end, $s = L$. Both the analytical and numerical solutions are divided by their maximum value, effectively removing $\tau_0$ as a variable. This procedure enables comparing their results in the five cases with different pull-out forces, amounts of pull-out, and sliding displacements.

As a demonstration of the stress field in the deformed composite, the results of case 2 are presented here. Fig. 16 shows the Mises stress field when all the nodes are sliding and the pull-out distance is 800 μm. As shown in Fig. 16(b), the fiber stress increases along fiber axial direction with the maximum value at the pulled end.

---

Table 3

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
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<td>$E_f$ (Pa)</td>
<td>$1E11$</td>
</tr>
<tr>
<td>$E_m$ (Pa)</td>
<td>$1E9$</td>
</tr>
<tr>
<td>Pressure applied (Pa)</td>
<td>$1E7$</td>
</tr>
</tbody>
</table>

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\textsuperscript{1} In order to generate an initial compressive stress at the interfaces, cases with a second set of boundary conditions where pressure was applied at both inner and outer matrix surfaces were performed. We found that this second set of boundary conditions delivered the same results as the first set of boundary conditions.
The profiles of the FE pressure distribution in the matrix nodes along the inner interface for all cases are shown in Fig. 17(a). In all cases, the interfacial pressure along the lower interface decreases near the fiber pulled end. The fiber bends upwards as it leaves the matrix, compressing the matrix along the upper interface and resulting in open nodes of no fiber/matrix contact along the lower interface and hence lowering the interfacial pressure near the fiber pulled end. Open nodes are also created at the other fiber end, as it slides out. At this gap, the applied pressure causes the matrix to push inwards, generating a high interfacial pressure there. Because of this, when comparing the results with the analytical model, we focus on the development of the interfacial pressure between the two fiber ends. Fig. 17(b) shows the same cases according to the analytical model. The model captures the same trends as seen in the FE simulations: the interfacial pressure increases at a faster rate for lower $R$ or higher $\mu$. Therefore our analytical model, although with the rigid matrix assumption, captures the trends of the interfacial shear stress distribution sufficiently.

4. Conclusions

In this paper, an analytical model for the pull-out curve associated with interfacial debonding and sliding in the pull-out process of a single curved fiber is developed. Two interface friction models are applied and compared. We show that a Coulomb friction model can better account for the fiber curvature effect than a constant friction model. A parametric study shows that fibers with more curvature, longer embedded lengths, and higher friction interfaces with the surrounding matrix require more force and energy for complete pull-out. Therefore, these conditions can help to effectively toughen the composites. Finally, a finite element model is built and the simulation results are compared with the corresponding analytical solutions. Reasonable agreement in the build up of the interfacial pressure is achieved in spite of the rigid pulley and rigid matrix assumptions made in the derivation. Our analytical solutions could be further used to interpret experimental data from pull-out tests to obtain interfacial properties of nanocomposites. This study revealed a few aspects which our model can be improved as discussed below.

First and most importantly, in the current derivation of our pull-out model, matrix deformation was neglected and had no effect on the interfacial shear stress distribution for debonding and sliding stages. However, it is expected that the Young’s modulus ratio of fiber and matrix has an effect on the interfacial shear stress distribution. More specifically, there should be a term of $E_f/E_m$ in the exponential function in Eq. (38) and (43). This can be achieved by incorporating matrix deformation into our current Coulomb friction formulation for the debonding and sliding stages. This would also improve the agreement between the FE and the analytical model, although Fig. 17 indicates our current analytical solution captures the trends of the interfacial shear stress distribution reasonably well. Furthermore, incorporating the matrix deformation into our formulation could also enable one to consider possible viscoelastic or other nonlinear deformation behavior of the polymer matrix.

Second, in our current pull-out model, fiber fracture is not considered. However, it is observed in nanocomposite experiments that some nanotubes are broken during pull-out tests (Duncan et al., 2007). A more comprehensive model would consider the fiber strength and account for the possibility of fiber fracture in different stages of pull-out.

Third, to further test the predictive capability of the model, it is desirable to perform finite element simulations for the entire pull-out process including bonded, debonding, and sliding stages. Finally, although the parametric study shows there is a potential for increased toughness with curved nanotubes, this pull-out model should be used in a fiber bridging crack model with curved fibers to address this concept more completely. This work is ongoing.

Acknowledgements

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Appendix. Single straight fiber pull-out for debonding and sliding stages with constant friction model

We begin with Lawrence’s analysis for a single straight fiber pull-out model with interfacial debonding (Lawrence, 1972), extend it to include sliding, and generate the entire pull-out curve. All equations in the following derivation are normalized to remove any unnecessary dependencies of the form of the solution on some parameters. The normalization factor for length is fiber radius $r_f$, that for stress and moduli is the fiber Young’s modulus $E_f$ and accordingly, that for force is $\pi(r_f^2)E_f$. The asterisks indicate normalized values.

A.1. Debonding stage

When the interfacial shear stress in the bonded stage increases up to the critical shear stress $\tau_s$, debonding begins. As shown in Fig. 1A, the fiber is then divided into two parts. One is the de-bonded part of the fiber $L-z$,
which suffers a constant interfacial shear stress $\tau_{ic}$, and the other part $z$ is still well-bonded to the matrix.

For the well-bonded part we use the previous solution for the interfacial shear stress of the bonded stage from (Chen et al., 2009):

$$\tau_i(z) = \frac{P^r\sqrt{Q} \cosh(\sqrt{Q}z)}{2} - \frac{\sigma_0\sqrt{Q}}{2 \sinh(\sqrt{Q}z)} = \tau_i^r.$$  \hspace{1cm} (2A)

Rearranging,

$$P^r = \frac{2\tau_i^r}{\sqrt{Q}} \tanh(\sqrt{Q}z) + \frac{\sigma_0}{\cosh(\sqrt{Q}z)}.$$  \hspace{1cm} (3A)

For the already debonded part, where $L^* - z^*$ is the debonded length, we have:

$$P^r - P^* = 2\tau_i^r(L^* - z^*),$$  \hspace{1cm} (4A)

$$\sigma_0^r(x') = P^r + 2\tau_i^r(x' - z^*), \hspace{1cm} x' \in (z^*, L^*).$$  \hspace{1cm} (5A)

Combining Eqs. (3A) and (4A), we have

$$P^r = 2\tau_i^r(L^* - z^*) + \frac{2\tau_i^r}{\sqrt{Q}} \tanh(\sqrt{Q}z^*)$$

$$\quad + \frac{\sigma_0}{\cosh(\sqrt{Q}z^*)}.$$  \hspace{1cm} (6A)

$P^r$ reaches either an extreme value or a saddle point when $\frac{dP^r}{dz} = 0$

$$\frac{dP^r}{dz} = -2\tau_i^r + 2\tau_i^r \log(\sqrt{Q}z^*),$$

$$\quad - \sqrt{Q}\sigma_0 \frac{\sinh(\sqrt{Q}z^*)}{\cosh^2(\sqrt{Q}z^*)} = 0.$$  \hspace{1cm} (7A)

where

$$z^* = \frac{1}{\sqrt{Q}} \sinh^{-1} \frac{\sigma_0^2 Q^* - 16\tau_i^r(\tau_i^r - \tau_i^r) - \sigma_0^2 \sqrt{Q}^*}{4\tau_i^r \sigma_0^2 Q^*}.$$  \hspace{1cm} (8A)

Checking $\frac{d^2P^r}{dz^2}$, we notice that when $z^* \to 0$, $\frac{d^2P^r}{dz^2} < 0$. We also notice that there is only one $z^*$ value. Thus, we can conclude that when $z^* = z^*$, $P^r = P^r_{max}$.

If $z^* > L^*$, debonding is catastrophic and the current pull-out force is large enough to debond the entire fiber. If, on the other hand $z^* < L^*$, progressive or stable debonding occurs and the pull-out force needs to be increased to further propagate the debond length. Correspondingly, we have

$$P_{max} = \begin{cases} 
2\tau_i^r(\sqrt{Q}z^*) + 2\tau_i^r \tanh(\sqrt{Q}z^*) + \frac{\sigma_0^r}{\cosh(\sqrt{Q}z^*)} & \text{if } z^* \geq L^* \\
2\tau_i^r(L^* - z^*) + \frac{2\tau_i^r}{\sqrt{Q}} \tanh(\sqrt{Q}z^*) + \frac{\sigma_0^r}{\cosh(\sqrt{Q}z^*)} & \text{if } z^* < L^*.
\end{cases}$$  \hspace{1cm} (9A)

Note that the fiber displacement can be written as a function of $z^*$ (Kerans and Patharsathathy, 1991):

$$\delta(\mathbf{II}) = \int_0^{z^*} dz^* + \int_{z^*}^{L^*} dz^* + P^r L^*$$

$$\quad = 1 \sqrt{Q} \left( p^r \cosh(\sqrt{Q}z^*) - p^r \right)$$

$$\quad - \frac{\sigma_0}{\sinh(\sqrt{Q}z^*)} + \frac{\sigma_0}{\cosh(\sqrt{Q}z^*)}$$

$$\quad + \int_{z^*}^{L^*} \left( p^r + 2\tau_i^r(x' - z^*) \right) dx^* + P^r L^*$$

$$\quad = \frac{2\tau_i^r}{\sqrt{Q}} \tanh(\sqrt{Q}z^*) + \frac{\sigma_0}{\cosh(\sqrt{Q}z^*)}$$

$$\quad + \frac{\sigma_0}{\cosh(\sqrt{Q}z^*)} - \frac{1}{\sinh(\sqrt{Q}z^*)}$$

$$\quad + (L^* - z^*) \left( 2\tau_i^r \tanh(\sqrt{Q}z^*) + \frac{\sigma_0}{\cosh(\sqrt{Q}z^*)} \right)$$

$$\quad + \tau_i^r(L^* - z^*)^2 + \frac{2\tau_i^r}{\sqrt{Q}} \tanh(\sqrt{Q}z^*)$$

$$\quad + \frac{\sigma_0}{\cosh(\sqrt{Q}z^*)} + 2\tau_i^r(L^* - z^*) \right).$$  \hspace{1cm} (10A)

![Fig. 3A. Stress equilibrium of the fiber in sliding, where $L^{(0)}$ is the embedded fiber length and $L - L^{(0)}$ is the rigid body displacement.](image-url)
Togetether with the already derived $P^e$ and $z^s$ relation, Eq. (6A), we can get the normalized $P - \delta$ curve for the debonding stage.

### 2. Sliding stage

After the entire fiber has debonded from the matrix, the sliding stage begins. Fig. 3A shows the stress equilibrium of the fiber. During sliding, the fiber undergoes elastic elongation and a rigid body displacement $L - L^{(III)}$, where $L^{(III)}$ is the remaining fiber length still embedded in the matrix.

From stress equilibrium of fiber in Fig. 3A, we have

$$\sigma^e_s (L^{(III)}) = P^e = 2\tau^e_s L^{(III)} \Rightarrow L^{(III)} = \frac{P^e}{2\tau^e_s}.$$  \hspace{1cm} (11A)

Then the fiber axial stress and axial strain are

$$\sigma^x (x') = 2\tau^x_s x' = P^x \frac{x'}{L^{(III)}},$$ \hspace{1cm} (12A)

$$\varepsilon^x (x') = 2\tau^x_s x', \quad x' \in (0, L^{(III)}).$$ \hspace{1cm} (13A)

The fiber displacement in sliding stage is composed of three parts: elastic elongation of the embedded part, elastic elongation of the fiber length originally extruded outside of the matrix, and the fiber rigid body sliding displacement (Kerans and Parthasarathy, 1991):

$$\delta^{(III)} = \int_{0}^{L^{(III)}} \varepsilon^x (x') dx' + P^x t_0 + L^* - L^{(III)} = \frac{\tau^x_s (L^{(III)})^2}{2} + P^x t_0 + L^* - L^{(III)} = \frac{P^x}{2\tau^x_s} + P^x t_0 + L^* - \frac{P^e}{2\tau^e_s}.$$ \hspace{1cm} (14A)

Based on the above normalized equations and that for the bonded stage in Appendix I in first paper (Chen et al., 2009), we obtain the entire normalized $P - \delta$ curves for single straight fiber pull-out as shown in Fig. 4A.

### References


